Epicycle approximation for close-to-circular orbits
Potential for close to circular orbits

• Circular orbit is the minimum of the effective potential, can Taylor expand around the minimum

\[
\Phi_{\text{eff}}(R, z; L_z) \approx \Phi_{\text{eff}}(R_g, 0; L_z) + \frac{1}{2} \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right)_{(R_g, 0)} (R - R_g)^2 + \frac{1}{2} \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \right)_{(R_g, 0)} z^2
\]

• Motion is then explicitly two decoupled oscillators

\[
\ddot{R} = \ddot{R} - \ddot{R}_g = - \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right)_{(R_g, 0)} (R - R_g),
\]

\[
\ddot{z} = - \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \right)_{(R_g, 0)} z,
\]
Frequencies

• Frequencies of the oscillations:

$$\kappa^2(R_g) = \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right) \bigg|_{(R_g,0)}$$

$$\nu^2(R_g) = \left( \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \right) \bigg|_{(R_g,0)}$$

• +azimuthal $\Omega = v_c(R) / R$

$$\frac{\kappa^2}{\Omega^2}(R_g) = 2 \left( \frac{d \ln[\Omega R^2]}{d \ln R} \right) \bigg|_{R_g}$$

• Range:

$$\Omega \lesssim \kappa \lesssim 2\Omega,$$
Vertical motion

- Solution is sinusoidal
- But bad approximation because disks are thin
- Vertical decoupling is useful, vertical epicycles not so much
Radial motion

• Radial oscillation around *guiding-center radius*: radius of circular orbit with angular momentum

\[ R(t) - R_g = X \cos(\kappa t + \alpha), \]

• Azimuthal motion from conservation of angular momentum

\[ \phi(t) = \phi_0 + \Omega t - 2 \frac{\Omega}{\kappa} \frac{X}{R_g} \sin(\kappa t + \alpha) \]

• Subtracting out motion of guiding center, motion is ellipse: epicycle

\[ R(t) - R_g = X \cos(\kappa t + \alpha), \]

\[ R_g (\phi(t) - \Omega t - \phi_0) = -2 \frac{\Omega}{\kappa} X \sin(\kappa t + \alpha) \]

• Axis ratio

\[ \gamma = \frac{2\Omega}{\kappa}, \quad \frac{1}{2} \lesssim \gamma \lesssim 1. \]
Epicycles near the Sun

- Measurements show that: \( \frac{\kappa(R_0)}{\Omega(R_0)} \approx 1.25 \)
- Vertical frequency \( \approx 2 \times \) radial frequency
- Intermediate age stars have epicycle amplitudes \( \approx 1.5 \) kpc
- Sun in `galpy`: `o = Orbit()`

http://galpy.readthedocs.io/en/latest/
Separability of disk orbits

• Beyond the epicycle approximation, orbits in disks are to a good approximation independent oscillations in $R$ and $z$ (coupling is small)

• Will be highly useful to understand equilibrium models of disks

• Write the potential as

$$\Phi(R, z) = \Phi_R(R) + \Phi(z; \tilde{R})$$

• Hamiltonian then splits into two pieces, with separately conserved energies

$$H_{\text{eff}}(R, z, p_R, p_z; L_z) = \frac{1}{2} \left( p_R^2 + +p_z^2 \right) + \Phi(R, z) + \frac{L_z^2}{2R^2}$$

$$= \frac{p_R^2}{2} + \Phi_R(R) + \frac{L_z^2}{2R^2} + \frac{p_z^2}{2} + \Phi_z(z; \tilde{R})$$

$$= H_{R, \text{eff}}(R, p_R) + H_z(z, p_z).$$
Separability of disk orbits?

Antoja et al. (2018; Gaia DR2)
Dynamical equilibrium recap
Galaxies are in a quasi-equilibrium state

- Galaxies reach quasi-steady-state on $O(t_{\text{dyn}})$ time scale
- Much happens, but quasi-equilibrium quickly restored
- Because dynamical times increases with increasing $r$, central regions much closer to equilibrium than outer regions
- Dynamical time clusters, outer halo: few Gyr $\rightarrow$ equilibrium suspect
Galaxy *disks* are in a quasi-equilibrium state

- Dynamical time for the planar motion in a galactic disk is ~few hundred Myr (e.g., near the Sun)

- Vertical oscillations are faster, $<\sim 100$ Myr $\rightarrow$ vertical structure equilibrates faster

- Expect galactic disks to be well-mixed dynamically
The *equilibrium* collisionless Boltzmann equation

- Collisionless Boltzmann equation (CBE) holds for any collisionless distribution function.

- For equilibrium system: \( f(x,v,t) = f(x,v) \) and the CBE becomes

\[
\dot{q} \frac{\partial f(q,p)}{\partial q} + \dot{p} \frac{\partial f(q,p)}{\partial p} = 0.
\]

- Fundamental equation of dynamical equilibria of galaxies.
Axisymmetric Jeans equations and the asymmetric drift
Spherical Jeans equations

\[ \frac{d(\nu \nu_r^2)}{dr} + \nu \left( \frac{d\Phi}{dr} + \frac{2v_r^2 - v_\theta^2 - v_\phi^2}{r} \right) = 0. \]

- in terms of \( \beta \)

\[ \frac{d(\nu \nu_r^2)}{dr} + 2\frac{\beta}{r} \nu v_r^2 = -\nu \frac{d\Phi}{dr}. \]

- In terms of enclosed mass

\[ M(< r) = -\frac{r \sigma_r^2}{G} \left( \frac{d \ln(\nu \sigma_r^2)}{d \ln r} + 2\beta \right) \]
Axisymmetric Jeans equations

• Start by writing down the collisionless Boltzmann equation in cylindrical coordinates; Hamiltonian is

\[
H = \frac{1}{2} \left( p_R^2 + \frac{p^2_\phi}{R^2} + p_z^2 \right) + \Phi(R, \phi, z)
\]

• And the collisionless Boltzmann equation is then

\[
p_R \frac{\partial f}{\partial R} + \frac{p_\phi}{R^2} \frac{\partial f}{\partial \phi} + p_z \frac{\partial f}{\partial z} + \left( \frac{p^2_\phi}{R^3} - \frac{\partial \Phi}{\partial R} \right) \frac{\partial f}{\partial p_R} - \frac{\partial \Phi}{\partial \phi} \frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0.
\]

• That looks complicated!

• For axisymmetric system, derivatives \(\text{wrt } \phi\) vanish

\[
p_R \frac{\partial f}{\partial R} + p_z \frac{\partial f}{\partial z} + \left( \frac{p^2_\phi}{R^3} - \frac{\partial \Phi}{\partial R} \right) \frac{\partial f}{\partial p_R} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial p_z} = 0.
\]
Axisymmetric Jeans equations

• Multiply by $p_R$ and integrate over all momenta

\[
\frac{\partial [\nu v_R^2]}{\partial R} + \frac{\partial [\nu v_R v_z]}{\partial z} + \nu \left( \frac{\partial \Phi}{\partial R} + \frac{v_R^2 - v_T^2}{R} \right) = 0.
\]

• This is the axisymmetric, radial Jeans equation

• Multiply by $p_z$ and integrate over all momenta

\[
\frac{\partial [\nu v_z^2]}{\partial z} + \frac{1}{R} \frac{\partial [R \nu v_R v_z]}{\partial R} + \nu \frac{\partial \Phi}{\partial z} = 0.
\]

• This is the axisymmetric, vertical Jeans equation
Axisymmetric Jeans equations for separable orbits

• If orbits separate into independent $R$ and $z$ motions, then the correlation between $v_R$ and $v_Z$ is zero.

• The Jeans equations then simplify to

$$\frac{\partial [\nu \sigma^2_R]}{\partial R} + \nu \left( \frac{\partial \Phi}{\partial R} + \frac{\sigma^2_R - \sigma^2_T - \overline{v_T}^2}{R} \right) = 0,$$

$$\frac{\partial [\nu \overline{v_Z}^2]}{\partial z} + \nu \frac{\partial \Phi}{\partial z} = 0.$$

• The vertical equation becomes similar to the spherical Jeans equation for vanishing anisotropy.

• An equilibrium vertical density is sustained by random motion —> dispersion supported.
Asymmetric drift

- Radial Jeans equation represents a balance between (a) the gravitational force, (b) mean motion around the center, and (c) random velocity (velocity dispersion).

- Let's re-write the radial equation to make this more clear at $z=0$, replacing the force with the circular velocity:

\[
\frac{v_c^2 - v_T^2}{\nu} = -\frac{R}{\nu} \frac{\partial [\nu \bar{v}_R^2]}{\partial R} - \frac{R}{\nu} \frac{\partial [\nu \bar{v}_R v_z]}{\partial z} + \sigma_T^2 - \bar{v}_R^2
\]

- or

\[
v_c - v_T = \frac{\bar{v}_R^2}{v_c + v_T} \left[ \frac{\sigma_T^2}{v_c^2} - 1 - \frac{\partial \ln [\nu \bar{v}_R^2]}{\partial \ln R} - \frac{R}{v_c^2} \frac{\partial \nu_R v_z}{\partial z} \right]
\]

- The square bracket is $O(1)$; for $\sigma_R << v_c$ therefore $v_c - <v_T> << v_c$, so we can write

\[
v_c - v_T = \frac{\bar{v}_R^2}{2v_c} \left[ \frac{\sigma_T^2}{v_c^2} - 1 - \frac{\partial \ln [\nu \bar{v}_R^2]}{\partial \ln R} - \frac{R}{v_c^2} \frac{\partial \nu_R v_z}{\partial z} \right]
\]
Asymmetric drift

\[ v_c - \bar{v}_T = \frac{v_R^2}{2v_c} \left[ \frac{\sigma_T^2}{v_R^2} - 1 - \frac{\partial \ln[\nu \bar{v}_R^2]}{\partial \ln R} - \frac{R}{v_R^2} \frac{\partial v_R \nu_z}{\partial z} \right] \]

• This relation is known as the Stromberg asymmetric drift relation

• Because the right-hand side does not vanish in general, it demonstrates that the average velocity of an equilibrium population of stars is not in general equal to the circular velocity —> we cannot simply assume that <v_T> = v_c

• The relation between <v_T> and v_c depends on
  • Radial density profile: d ln nu / d ln R
  • The velocity dispersion \( \sigma_R \) and its radial profile
  • The ratio of the tangential and radial velocity dispersion

• Sign of square bracket is typically positive and therefore <v_T> is smaller than v_c; this is the normally assumed behavior

• But the sign can be negative as well, and then <v_T> is larger than v_c!
Distributions functions for thin disks
Distribution functions for disks and the Jeans theorem

- Week 3: Jeans theorem: equilibrium DF is function of integrals of motion
  \[ f = f(I) \]

- For separable orbits, we have three integrals: \( L_z, E_R, \) and \( E_z \) \( \rightarrow \) \( f = f(L_z, E_R, E_z) \)

- Because of separability, we can assume that the distribution function separates

  \[ f(L_z, E_R, E_z) = f(L_z, E_R) \times f(E_z|L_z) \]

  First factor is the \textit{planar} part of the DF, second factor is a vertical part at a given \( L_z \) (or, guiding-center radius \( R_g \times v_c = L_z \))

- Thus, we can build simple equilibrium models by combining simple planar and vertical DFs
Distribution function for a cold, razor-thin disk

- Cold, razor-thin disk = all orbits are circular, with some surface density profile $\Sigma(R) = \Sigma(L_z)$

- Distribution function must therefore look like this:

$$f(E, L_z) = F(L_z) \delta(E - E_c[L_z])$$

- $E_c[L_z]$ is the energy of a circular orbit with the given $L_z$

- with $F(L_z)$ determined by $\Sigma(R)$

- We want to integrate this DF over velocity $(v_R, v_T)$ at a given $R$ and then match $F(L_z)$ to $\Sigma(R)$

$$f(E, L_z) = \frac{\gamma(R_L) \Sigma(R_L)}{2\pi} \delta(E - E_c[L_z]).$$
Distribution functions for a warm, razor-thin disk

• A ‘warm’ disk has orbits that are non-circular

• We can build such a disk by warming up the cold disk DF

• We do this by replacing the $\delta(E-E_c[L_z])$ with a finite-width kernel

$$\delta(E - E_c[L_z]) \to K[(E - E_c[L_z])/\sigma_R^2(R)].$$

• We have a lot of freedom in this choice!

• In general, this will lead to $\Sigma'(R) =/= \Sigma(R)$, but for small dispersion typically $\Sigma'(R) \sim \Sigma(R)$

• Either just live with this, or can adjust the DF's pre-factor
The Schwarzschild DF

• One popular choice is to replace

$$\delta(E-E_c[L_z]) \rightarrow \exp([E-E_c[L_z]]/\sigma_R^2)$$

which gives the Shu DF:

$$f(E, L_z) = \frac{\gamma(R_L) \Sigma(R_L)}{2\pi} \frac{1}{\sigma_R^2(R_L)} \exp\left(-\frac{E - E_c[L_z]}{\sigma_R^2(R_L)}\right).$$

• If we use the epicycle approximation to replace $E-E_c[L_z]$, we get the Schwarzschild DF:

$$f(E, L_z) = \frac{\gamma(R_L) \Sigma(R_L)}{2\pi \sigma_R^2(R_L)} \exp\left(-\frac{v_R^2 + \gamma^2 (v_T - v_c)^2}{2\sigma_R^2(R_L)}\right).$$
The Schwarzschild DF

- For close-to-circular orbits, $\Sigma(R)$ and $\sigma_R(R) \sim$ constant

- The velocity distribution is then a Gaussian with $<v_R> = 0$, $<v_T> = v_c$, and $\sigma'_T(R) \approx \sigma'_R(R)/\gamma$

- For ‘warmer’ distribution functions, the velocity distribution becomes non-Gaussian
Asymmetric drift re-visited

• Previous velocity distribution demonstrates that \(<v_T>\) less than \(v_c\) when \(\sigma_R\) increases

• This is the asymmetric drift

• Physically, at a given radius \(R\) we see stars coming from radii \(< R\) and radii \(> R\); for \(v_c(R) \sim \text{flat}\)
  • Those with radii \(< R\) are on the outer part of their orbits
    \(\rightarrow v_T \text{ less than } v_c\)
  • Those with radii \(> R\) are on the inner part of their orbits
    \(\rightarrow v_T \text{ greater than } v_c\)

• For a declining surface density: there are more stars with radii \(< R\) than there are stars with radii \(> R\) \(\rightarrow\) mean effect is for \(<v_T>\) to be less than \(v_c\)

• Exacerbated by declining \(\sigma_R\) \(\rightarrow\) stars with radii \(< R\) can be coming from further away

• But if the density gradient is different, can get the opposite effect! Often overlooked!
‘Reverse’ asymmetric drift

\[ \Sigma(R) = \Sigma_0 e^{R_0/3} \]

\[ \sigma_0 = 0.02 v_c \quad \sigma_0 = 0.07 v_c \quad \sigma_0 = 0.15 v_c \]
The velocity distribution in the solar neighborhood
Velocities in the solar neighborhood

• We can measure the velocities for large samples of stars in the solar neighborhood (e.g., Hipparcos, now Gaia)

• We can investigate these with the tools that we have discussed so far this week

• First we can correct the observed motion wrt the Sun for the (small) effect of Galactic rotation using the Oort constants, because the Oort constants give the first-order effect of Galactic rotation wrt distance from the Sun (this is a small effect for stars <~ few 100 pc)

\[
\begin{align*}
\nu_{\text{los}}(D, l) &\rightarrow \nu_{\text{los}}(D, l) - D \left[ K + A \sin 2l + C \cos 2l \right] \cos b , \\
\mu_l(D, l) &\rightarrow \mu_l(D, l) - (B + A \cos 2l - C \sin 2l) \cos b , \\
\mu_b(D, l) &\rightarrow \mu_b(D, l) + (K + A \sin 2l + C \cos 2l) \sin b \cos b .
\end{align*}
\]
The solar motion

• We measure velocities (U,V,W) wrt the Sun, but the Sun’s velocity itself wrt a circular orbit is not well known

• But we can use the Stromberg asymmetric drift relation to figure out the Sun’s motion!

• If we don’t know the Sun’s motion wrt a circular orbit in the direction of Galactic rotation, but label it as V_0, we get

\[-\bar{V} - V_0 = \frac{\sigma_U^2}{2v_c} \left[ \frac{\sigma_V^2}{\sigma_U^2} - 1 - \frac{\partial \ln[v \sigma_U^2]}{\partial \ln R} - \frac{R}{\sigma_U^2} \frac{\partial \sigma_{UW}^2}{\partial z} \right] .\]

• If we can apply this equation for a population with small dispersion \(\sigma_U\) or extrapolate from populations with larger \(\sigma_U\) to zero, then we can read off the Solar motion as minus the mean velocity of this population
The solar motion

\[ -\bar{V} - V_0 = \frac{\sigma_U^2}{2v_c} \left[ \frac{\sigma_V^2}{\sigma_U^2} - 1 - \frac{\partial \ln[\nu \sigma_U^2]}{\partial \ln R} - \frac{R}{\sigma_U^2} \frac{\partial \sigma_{UW}^2}{\partial z} \right] . \]

• Applying this equation for a single population is difficult: we need to measure all quantities in square brackets and radial gradients are difficult to measure from local measurements.

• Historically, assumption was that the factor in square brackets does not depend on \( \sigma_U \).

• If we can measure the mean velocity wrt Sun and \( \sigma_U^2 \), then these quantities should follow a straight line and \( V_0 = \) intercept.
The solar motion with *Hipparcos*

- Dehnen & Binney (1998) attempted this with *Hipparcos* data
- Measured mean velocities and velocity dispersion for populations along the main sequence
The solar motion with *Hipparcos*

\[ U_0 = 10.00 \pm 0.36 \text{ km s}^{-1}, \quad W_0 = 7.17 \pm 0.38 \text{ km s}^{-1}, \quad V_0 = 5.25 \pm 0.62 \text{ km s}^{-1}. \]
Why would the linear asymmetric drift analysis fail

- Linear $\sigma_U^2$ vs. $<V>$ trend assumes that the square bracket in the asymmetric drift equation is the same for all populations of stars

- Crucially, this includes the radial profile

- But studies have shown that the radial profile of different stellar populations in the Milky Way are very different
Why would the linear asymmetric drift analysis fail

- Radial profile *not* exponential and strong trends with age and abundance

- If there is a correlation between scale length and population—which is very plausible when binning in color along the main-sequence—then the linear analysis will fail
Why would the linear asymmetric drift analysis fail

- Schonrich et al. (2010) used a model for stellar populations in the Milky Way and performed a mock measurement of the asymmetric drift (green squares) —>

- It is clear that the linear trend does not continue at low $\sigma_U^2$

Schonrich et al. (2010)

$$V_0 = 12.24 \pm 0.47 \text{ stat.} \pm 2 \text{ (syst.) km s}^{-1}$$
Why would the linear asymmetric drift analysis fail

- We can understand this using our simple disk DF models for warm disks
- Blue points all have the same radial profile —> linear analysis works
- Orange points have declining radial profile at large $\sigma_U^2$ that becomes flat around $\sigma_U^2 \sim 500 \text{ km}^2/\text{s}^2$ and becomes increasing for lower $\sigma_U^2$ —> $\langle V \rangle$ becomes $\sim$ constant
Full two-dimensional velocity distribution

- Warm disk DFs also predict the full in-plane velocity distribution
- Schwarzschild: Gaussian velocity distribution
- Better DFs: Gaussian-ish, but smooth
Observed two-dimensional velocity distribution

Dehnen (1998)
Observed two-dimensional velocity distribution

- Observed velocity distribution is far from smooth, but is characterized by a large number of clumps: *moving groups*
- About 40% of all stars are part of the clumps
- Contain stars with a wide range in age and abundances
- Likely caused by dynamical interactions with bar, spiral structure

Bovy et al. (2009)
The vertical equilibrium of disk
Vertical equilibrium

- So far discussed distribution in \((R, v_R, v_T)\)
- Vertically, stars oscillate around the \(z=0\) mid-plane
- Approximately decoupled from the in-plane motion
- In this case, we can write down a 1D collisionless Boltzmann equation

\[

v_z \frac{\partial f}{\partial z} - \frac{d\Phi_z(z)}{dz} \frac{\partial f}{\partial v_z} = 0.

\]
Vertical dynamics

• Decoupling goes one step further, because we can also approximate the Poisson equation as a 1D equation:

\[
4\pi G \rho = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) + \frac{\partial^2 \Phi}{\partial z^2}
\]

\[
\approx \frac{1}{R} \frac{d v_c^2}{dR} + \frac{\partial^2 \Phi}{\partial z^2}.
\]

• First term is approx. constant up to few kpc above the disk (Bovy & Tremaine 2012) —> can write it as a phantom density

\[
4\pi G \rho_{RC}(R) = \frac{1}{R} \frac{d v_c^2}{dR}
\]

• and the Poisson equation becomes

\[
4\pi G [\rho(R, z) - \rho_{RC}(R)] = \frac{d^2 \Phi}{dz^2}
\]

• Thus, when we constrain the vertical potential using stars, we directly measure the local density
Self-gravitating disk

\[ v_z \frac{\partial f}{\partial z} - \frac{d\Phi_z(z)}{dz} \frac{\partial f}{\partial v_z} = 0. \]

- Consider a solution of the following form
  \[ f(E_z) = \frac{\rho_0}{\sqrt{2\pi}\sigma_z} e^{-\frac{E_z}{\sigma_z^2}}. \]
- Like the isothermal sphere, also similar to disk DFs
- Velocity distribution is Gaussian:
  \[ f(v_z|z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\frac{v_z^2}{\sigma_z^2}}. \]
- Density is then:
  \[ \nu(z) = \rho_0 e^{-\frac{\Phi_z(z)}{\sigma_z}}. \]
- In the Poisson equation it goes:
  \[ \frac{d^2\Phi_z}{dz^2} = 4\pi G \rho_0 e^{-\frac{\Phi_z(z)}{\sigma_z}}. \]
Self-gravitating disk

- Solution:

\[ \Phi_z(z) = 2 \sigma^2 \ln\left( \cosh\left[ \frac{x}{2H} \right] \right), \]
\[ \rho(z) = \rho_0 \text{sech}^2 \left[ \frac{x}{2H} \right], \]
\[ H^2 = \sigma^2 / [8\pi G \rho_0] \]
The vertical Jeans equation
The vertical Jeans equation

- From before:

\[- \frac{\partial \Phi}{\partial z} = \frac{1}{\nu} \frac{\partial [\nu \sigma_z^2]}{\partial z} + \frac{1}{R \nu} \frac{\partial [R \nu \sigma_{Rz}^2]}{\partial R}\]

- Second term: *tilt of the velocity ellipsoid*, zero when planar and vertical motion is decoupled

- We can write this term in terms of a tilt angle: \( \tan(2 \alpha) = 2 \frac{\sigma_{Rz}^2}{\sigma_R^2 - \sigma_z^2} \).

- Extreme possibilities:
  - decoupled: \( \alpha = 0 \)
  - Spherical system: velocity ellipsoid aligned with radial direction: *points toward the center*: \( \tan \alpha = z/R \)

- Close to the disk, tilt is probably somewhere in between
- Contribution from the tilt to the Jeans equation: small fraction of the main terms
The vertical Jeans equation and the vertical Poisson equation

• When we integrate the vertical Poisson equation between -z and z:

\[
\Sigma(z; R) = \frac{1}{2\pi G} \left[ \frac{\partial \Phi}{\partial z} + \frac{1}{R} \int_0^\infty dz \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) \right]
\]

• Substitute the vertical Jeans equation and the phantom density

\[
\Sigma(z; R) = \frac{1}{2\pi G} \left[ z \left( B^2 - A^2 \right) - \frac{1}{\nu} \frac{\partial [\nu \sigma_z^2]}{\partial z} - \frac{1}{R \nu} \frac{\partial [R \nu \sigma_{Rz}^2]}{\partial R} \right].
\]

• ——> measuring the vertical density and kinematics
  ~directly measures the surface density
Vertical equilibrium near the Sun
Oort limit

• Very close to the Sun (z <~ 100 pc) the density of all matter is ~constant $\rho_0$

• Poisson equation is then solved by

$$\Phi_z(z) = \frac{\omega^2 z^2}{2}$$

$$\omega^2 = 4\pi G [\rho_0 + \rho_{RC}(R)]$$,

• Or we can measure $\Sigma(z;R)$ and get $\rho_0$ from the slope of $\Sigma(z;R)$ vs z.

• First done by Oort (1932)

• First use of the term “dark matter” to explain discrepancy between dynamically-inferred $\rho_0$ and directly measured $\rho_0$
Oort limit  Oort (1932)

\[ K(z) \times 10^9 \]

Force in \(10^{-9} \text{ cm/s}^2\)

Height in pc

\[
10^{-9} \text{ cm s}^{-2} \approx 0.3 \left(\text{km s}^{-1}\right)^2 \text{ pc}^{-1},
\]

\[
\left(\text{km s}^{-1}\right)^2 \text{ pc}^{-1}/[2\pi G] \approx 37 M_\odot \text{ pc}^{-2}.
\]

\[
\rho_0 \approx (3.5 \text{ cm s}^{-2}/[200 \text{ pc}]/2) \approx 0.1 M_\odot \text{ pc}^{-3}.
\]
Oort limit: best current measurement

- Holmberg & Flynn (2000): A and F-type dwarfs from *Hipparcos*

- Measure their local velocity distribution $\rightarrow$ DF inversion to go to density

$$
\nu(z) = 2 \int_{\sqrt{2 \Phi_z(z)}}^{\infty} \frac{d\nu_z}{\sqrt{\nu_z^2 - 2 \Phi_z(z)}} \frac{v_z f(v_z | z = 0)}{\sqrt{\nu_z^2 - 2 \Phi_z(z)}}
$$

- Compare to observed density for different potentials $\rightarrow$ constrain $\rho_0$

$$
\rho_0 = 0.102 \pm 0.010 \, M_\odot \, \text{pc}^{-3}
$$

Holmberg & Flynn (2000)
Jeans analysis at larger heights

- Jeans analysis allows us to measure $\Sigma(z; R)$ as a function of $z$

- Disk mass contained at $|z| <\sim 1$ kpc

- Can read off dark-matter density from slope at $|z| > 1$ kpc

- and can then measure disk density as the rest
Jeans analysis at larger heights

\[
\rho_{DM} = 0.008 \pm 0.003 \, M_\odot \, \text{pc}^{-2}.
\]

Bovy & Tremaine (2012)