AST1420 "Galactic Structure and Dynamics"
Problem Set 1

Due on Oct. 15 at the start of class

Most of the exercises in this problem set must be solved on a computer and the best way to hand in the problem set is as an `ipython notebook`. Rather than sending me the notebook, you can upload it to `GitHub`, which will automatically render the notebook. Rather than starting a repository for a single notebook, you can upload your notebook as a `gist`, which are version-controlled snippets of code that can optionally be made private. If you want to make further changes, you can clone your gist in a separate directory and use it as you would any other git repository. *Please re-run the entire notebook (with* `Cell > Run All`*) after re-starting the notebook kernel before uploading it*; this will make sure that the input and output are fully consistent.

If you are unfamiliar with notebooks, you can also send in a traditional write-up (in LaTeX), but you also need to send in well-commented code for how you solved the problems. Thus, notebooks are strongly preferred :-)

Problem 1: The virial mass of the NFW profile. The virial mass of a dark matter halo is arguably its most fundamental parameter, because the tight correlation between correlation and mass found in numerical simulations of dark-matter halo formation means that dark-matter halos in nature form an essentially one-dimensional sequence of mass. However, the virial mass depends on how one chooses the overdensity Δ_v that defines the virial radius. A standard value for this is $\Delta_v = 200$, but as we will see when we discuss the formation of dark-matter halos in more detail, Δ_v should depend on the cosmological parameters and the redshift of the halo's formation. When doing this, $\Delta_v \approx 200$ at high redshift ($z \gtrsim 2$) in our Universe, but at the present day a value of $\Delta_v \approx 100$ is more correct. For a quantity as fundamental as the virial mass, many discussions of it in papers and elsewhere are surprisingly vague on the overdensity used to define it! Let's see how much of an issue this is.

(a) Using the equations given for the NFW profile and using values appropriate for the Milky Way's dark-matter halo ($\rho_0 = 0.0035\,M_\odot\,\mathrm{pc}^{-3}$ and $a = 16\,\mathrm{kpc}$), compute the virial radius and virial mass as a function of Δ_v and plot them. Discuss how the virial radius and virial mass depend on Δ_v. Use $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

(b) What about the NFW density profile causes the behavior that you see?

Problem 2: In his colloquium a few weeks ago, Scott Tremaine discussed scattering of comets by the planets in the solar system as the comets pass through the inner solar system. He mentioned that scattering of the comets tends to preserve their pericentric distances. Let's understand why that is using what we know about orbits in spherical potentials!

(a) Consider an orbit in a spherical isochrone potential with $b = 1$ (pick an orbit that explores $r \approx 1$ and is not too close to circular). Using orbit integration in `galpy`, add an instantaneous velocity offset when the orbit is at its pericenter radius. Investigate what happens to the pericenter radius of the resulting orbit. Is it larger or smaller than the original pericenter radius? It is useful to consider the special cases where (i) you only change the radial velocity and (ii) you only change the tangential velocity (or equivalently the angular momentum). Does the answer change if you consider different orbits?

(b) Argue why the behavior you saw in part (a) is true for *any* orbit in *any* spherical potential. (Hint: consider the special cases and what happens to the effective potential). You can illustrate your argument with the orbit(s) that you investigated in (a), but make it clear why the behavior is general.

(c) What happens to the apocenter radius and the eccentricity in (a)? Investigate numerically and explain what is happening.

(d) Now consider the orbit of a comet that originates from the Oort cloud, at 20,000 AU and has an orbit that brings it to 2 AU. The perturbations to the orbit near its pericenter from the planets lead to changes in the energy that are equivalent to changing the inverse semi-major axis by $10^{-4}\,\mathrm{AU}^{-1}$. By considering a few different ways of distributing this energy change into radial and tangential velocity kicks, determine how the pericenter and apocenter distances of this comet change. Do you see what Scott claimed? Discuss.

Problem 3: The cored isothermal sphere and self-interacting dark matter models.

(a) Equation (6.71) for the density of an isothermal sphere has non-singular solutions that can be found by specifying the boundary condition for a core: $\rho(0) = \rho_0$ and $\mathrm{d}\rho/\mathrm{d}r = 0$ at $r = 0$. Demonstrate by writing Equation (6.71) in terms of $y = \ln \tilde{\rho}/\rho_0$ and $x = r/r_0$ where $r_0^2 = 9\sigma^2/[4\pi G\rho_0]$ that cored solutions have the form $\rho(r) = \rho_0\, f(r/r_0)$ and give the equation that determines $f(x)$.

(b) Write a function that computes $f(x)$ and use it to plot ρ/ρ_0 as a function of r/r_0 for the cored isothermal sphere. Compare what you see to the singular isothermal sphere.

(c) An often preferred model for the dark matter density profile is the NFW profile. Using an NFW profile with concentration 11.5 and a virial mass of $7 \times 10^{11}\, M_\odot$ (for $\Delta_v = 200$; this is like the Milky Way's dark matter halo), numerically compute the radial velocity dispersion profile in $\mathrm{km\,s^{-1}}$ for $\beta = 0$ and $\beta = 0.5$ (implement the integrals yourself, don't just use `galpy.df.jeans`). Plot your solution on a logarithmic grid from $r = 1\,\mathrm{kpc}$ to $r = 300\,\mathrm{kpc}$.

(d) The cored isothermal sphere describes the inner regions of dark matter halos in models where dark matter particles interact strongly enough that they scatter off of each other and thermalize in regions of high enough dark-matter density (through interactions that are analogous to non-gravitational interactions between baryons). This thermalization homogenizes the velocity dispersion and this means that in these models, the outer dark matter profile is given by the standard NFW form, while the inner profile is that of the cored isothermal sphere. The boundary between these two regimes is at the radius where a dark matter particle is expected to scatter once. The scattering rate per unit time per particle is given by

$$\Gamma(r) = \frac{\sigma}{m}\,\frac{4}{\sqrt{\pi}}\,\sigma_r(r)\,\rho(r)\,, \tag{1}$$

where σ/m is the self-interacting dark matter cross section per unit mass (note that this is *not* the same σ as that of the isothermal sphere, but σ/m is the standard notation for this cross section), $\sigma_r(r)$ is the radial velocity dispersion, and $\rho(r)$ the density profile. For $\sigma/m = 1\,\mathrm{cm}^2\,\mathrm{g}^{-1}$, a halo age of 10 Gyr, and the NFW halo and radial-velocity dispersion profiles from (c), determine the radius r_1 at which a particle in the NFW halo is expected to scatter once, both for $\beta = 0$ and $\beta = 0.5$.

(e) Given the cored-isothermal sphere profile that you found in (b), find the cored-isothermal profile (that is, the parameters ρ_0 and r_0) such that the cored-isothermal profile's density and enclosed mass matches that of the NFW profile from (c) and (d) at r_1 (for the $\beta = 0$ case for the NFW's velocity dispersion). Plot the entire density profile of the cored-isothermal profile out to r_1 and the NFW profile outside of that from 1 kpc to 300 kpc. This is a simple model for the Milky Way's dark matter halo if dark matter has strong self interactions!